

ABS ALGORITHMS FOR LINEAR EQUATIONS AND ABSPACK

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ABSTRACT - We present the main results obtained during a research on ABS methods in the framework of the project Analisi Numerica e Matematica Computazionale.

1. – The ABS algorithms

ABS algorithms were introduced by Abaffy, Broyden and Spedicato (1984), to solve linear equations first in the form of the *basic ABS class*, later generalized as the *scaled ABS class* and applied to linear least squares, nonlinear equations and optimization problems, see e.g. the monographs by Abaffy and Spedicato (1989) and Zhang, Xia and Feng (1999), or the bibliography by Nicolai and Spedicato (1997) listing over 300 ABS papers. In this paper we consider some new results obtained in the framework of project Analisi Numerica e Matematica Computazionale, including the performance of several codes of ABSPACK, a FORTRAN package under development.

For later reference, we recall the scaled ABS algorithm for solving the following determined or underdetermined linear system, where $rank(A)$ is arbitrary and $A^T = (a_1, \dots, a_m)$

$$Ax = b \quad x \in R^n, \quad b \in R^m, \quad m \leq n \quad (1)$$

or

$$a_i^T x - b_i = 0, \quad i = 1, \dots, m \quad (2)$$

The scaled ABS algorithm

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- (A) Give $x_1 \in R^n$ arbitrary, $H_1 \in R^{n,n}$ nonsingular arbitrary, $v_1 \in R^m$ arbitrary nonzero. Set $i = 1$.
- (B) Compute the residual $r_i = Ax_i - b$. If $r_i = 0$ stop (x_i solves the problem.) Otherwise compute $s_i = H_i A^T v_i$. If $s_i \neq 0$, then go to (C). If $s_i = 0$ and $\tau = v_i^T r_i = 0$, then set $x_{i+1} = x_i$, $H_{i+1} = H_i$ (the i -th equation is redundant) and go to (F). Otherwise stop (the system has no solution).
- (C) Compute the search vector p_i by

$$p_i = H_i^T z_i \quad (3)$$

where $z_i \in R^n$ is arbitrary save for the condition

$$v_i^T A H_i^T z_i \neq 0 \quad (4)$$

- (D) Update the estimate of the solution by

$$x_{i+1} = x_i - \alpha_i p_i, \quad \alpha_i = v_i^T r_i / v_i^T A p_i. \quad (5)$$

- (E) Update the matrix H_i by

$$H_{i+1} = H_i - H_i A^T v_i w_i^T H_i / w_i^T H_i A^T v_i \quad (6)$$

where $w_i \in R^n$ is arbitrary save for the condition

$$w_i^T H_i A^T v_i \neq 0. \quad (7)$$

- (F) If $i = m$, stop (x_{m+1} solves the system). Otherwise give $v_{i+1} \in R^m$ arbitrary linearly independent from v_1, \dots, v_i . Increment i by one and go to (B).

Matrices H_i , which are generalizations of (oblique) projection matrices, have been named *Abaffians* at the First International Conference on ABS methods (Luoyang, China, 1991). There are alternative formulations of the scaled ABS algorithms, e.g. using vectors instead of the square matrix H_i , with possible advantages in storage and number of operations. In a next section we will show how they can be used to generate infinite iterative methods.

The choice of the parameters H_1 , v_i , z_i , w_i determines particular methods. The *basic ABS class* is obtained by taking $v_i = e_i$, as the i -th unit vector in R^m .

We recall some properties of the scaled ABS class, assuming that A has full rank.

- Define $V_i = (v_1, \dots, v_i)$ and $W_i = (w_1, \dots, w_i)$. Then $H_{i+1} A^T V_i = 0$ and $H_{i+1}^T W_i = 0$.

- The vectors $H_i A^T v_i$, $H_i^T w_i$ are zero if and only if a_i , w_i are respectively linearly dependent from a_1, \dots, a_{i-1} , w_1, \dots, w_{i-1} .
- Define $P_i = (p_1, \dots, p_i)$ and $A_i = (a_1, \dots, a_i)$. Then the implicit factorization $V_i^T A_i^T P_i = L_i$ holds, where L_i is nonsingular lower triangular. Hence, if $m = n$, one obtains a semiexplicit factorization of the inverse, with $P = P_n$, $V = V_n$, $L = L_n$

$$A^{-1} = PL^{-1}V^T. \quad (8)$$

For several choices of V the matrix L is diagonal, hence formula (8) gives a fully explicit factorization of the inverse as a byproduct of the ABS solution of a linear system.

- The general solution of system (1) can be written as follows, with $q \in R^n$ arbitrary

$$x = x_{m+1} + H_{m+1}^T q \quad (10)$$

- The Abaffian can be written in term of the parameter matrices as

$$H_{i+1} = H_1 - H_1 A^T V_i (W_i^T H_1 A^T V_i)^{-1} W_i^T H_1. \quad (11)$$

Letting $V = V_m$, $W = W_m$, one can show that the parameter matrices H_1 , V , W are admissible (i.e. condition (7) is satisfied) iff the matrix $Q = V^T A H_1^T W$ is strongly nonsingular (i.e. it is LU factorizable). This condition can be satisfied by exchanges of the columns of V or W . If Q is strongly nonsingular and we take, as is done in all algorithms so far considered, $z_i = w_i$, then condition (4) is also satisfied. Analysis of the conditions under which Q is not strongly nonsingular leads, when dealing with Krylov space methods in their ABS formulation, to a characterization of the topology of the starting points that can produce a breakdown (either a division by zero or a vanishing search direction) and to several ways of curing it, including those considered in the literature.

Two subclasses of the scaled ABS class and particular algorithms are now recalled.

- (a) The *conjugate direction subclass*. This class is obtained by setting $v_i = p_i$. It is well defined under the condition (sufficient but not necessary) that A is symmetric and positive definite. It contains the ABS versions of the Choleski, the Hestenes-Stiefel and the Lanczos algorithms. This class generates all possible algorithms whose search directions are A -conjugate. If $x_1 = 0$, the vector x_{i+1} minimizes the energy (A -weighted Euclidean) norm of the error over $Span(p_1, \dots, p_i)$ and the solution is approached monotonically from below in the energy norm.

- (b) The *orthogonally scaled subclass*. This class is obtained by setting $v_i = Ap_i$. It is well defined if A has full column rank and remains well defined even if m is greater than n . It contains the ABS formulation of the QR algorithm (the so called *implicit QR algorithm*), the GMRES and the conjugate residual algorithms. The scaling vectors are orthogonal and the search vectors are $A^T A$ -conjugate. If $x_1 = 0$, the vector x_{i+1} minimizes the Euclidean norm of the residual over $\text{Span}(p_1, \dots, p_i)$ and the solution is monotonically approached from below in the residual norm. It can be shown that the methods in this class can be applied to overdetermined systems of $m > n$ equations, where in n steps they obtain the solution in the least squares sense.
- (c) The *optimally stable subclass*. This class is obtained by setting $v_i = A^{-T} p_i$, the inverse disappearing in the actual recursions. The search vectors in this class are orthogonal. If $x_1 = 0$, then the vector x_{i+1} is the vector of least Euclidean norm over $\text{Span}(p_1, \dots, p_i)$ and the solution is approached monotonically from below in the Euclidean norm. The methods of Gram-Schmidt and of Craig belong to this subclass. The methods in this class have minimum error growth in the approximation to the solution according to a criterion by Broyden.
- (d) The *Huang algorithm* is obtained by the choices $H_1 = I$, $z_i = w_i = a_i$, $v_i = e_i$. A mathematically equivalent, but numerically more stable, formulation is the so called *modified Huang algorithm* ($p_i = H_i(H_i a_i)$ and $H_{i+1} = H_i - p_i p_i^T / p_i^T p_i$). Huang algorithm generates search vectors that are orthogonal and identical with those obtained by the Gram-Schmidt procedure applied to the rows of A . If $x_1 = 0$, then x_{i+1} is the solution with least Euclidean norm of the first i equations. The solution x^+ with least Euclidean norm of the whole system is approached monotonically and from below by the sequence x_i .
- (e) The *implicit LU algorithm* is given by the choices $H_1 = I$, $z_i = w_i = v_i = e_i$. It is well defined iff A is regular (i.e. all principal submatrices are nonsingular). Otherwise column pivoting has to be performed (or, if $m = n$, equation pivoting). The Abaffian has the following structure, with $K_i \in R^{n-i, i}$

$$H_{i+1} = \begin{bmatrix} 0 & 0 \\ \cdots & \cdots \\ 0 & 0 \\ K_i & I_{n-i} \end{bmatrix}. \quad (12)$$

implying that the matrix P_i is unit upper triangular, so that the implicit factorization $A = LP^{-1}$ is of the LU type, with units on the diagonal. The algorithm requires for $m = n$, $n^3/3$ multiplications plus lower order terms, the same cost of classical LU factorization or Gaussian elimination.

Storage requirement for K_i requires at most $n^2/4$ positions, i.e. half the storage needed by Gaussian elimination and a fourth that needed by the LU factorization algorithm (assuming that A is not overwritten). Hence the implicit LU algorithm has same arithmetic cost but uses less memory than the most efficient classical methods.

- (f) The *implicit LX algorithm*, see Spedicato, Xia and Zhang (1997), is defined by the choices $H_1 = I, v_i = e_i, z_i = w_i = e_{k_i}$, where k_i is an integer, $1 \leq k_i \leq n$, such that $e_{k_i}^T H_i a_i \neq 0$. By a general property of the ABS class for A with full rank there is at least one index k_i such that $e_{k_i}^T H_i a_i \neq 0$. For stability reasons we select k_i such that $|e_{k_i}^T H_i a_i|$ is maximized. This algorithm has the same overhead and memory requirement as the implicit LU algorithm, but does not require pivoting. Its computational performance is also superior and generally better than the performance of the classical LU factorization algorithm with row pivoting, as available for instance in LAPACK or MATLAB, see Mirnia (1996). Therefore this algorithm can be considered as *the most efficient general purpose linear solver not of the Strassen type*. The implicit LX algorithm has also very important applications in a reformulation of the simplex method for the LP problem, see Zhang, Xia and Feng (1999), where it leads to a reduction of storage up to a factor 8 and of multiplications up to one order for problems where there is a small number of degrees of freedom, with respect to implementations based upon the classical LU factorization.

2. – Solution of linear Diophantine equations

One of our main results has been the derivation of ABS methods for linear Diophantine equations. The ABS algorithm determines if the Diophantine system has an integer solution, computes a particular solution and provides a representation of all integer solutions. It is a generalization of a method proposed by Egervary (1955) for the particular case of a homogeneous system.

Let Z be the set of all integers and consider the Diophantine linear system of equations

$$Ax = b, \quad x \in Z^n, \quad A \in Z^{m \times n}, \quad b \in Z^m, \quad m \leq n. \quad (13)$$

It is intriguing that while thousands of papers have been written concerning nonlinear, usually polynomial, Diophantine equations in few variables, the general linear system has attracted much less attention. The single linear equation in n variables was first solved by Bertrand and Betti (1850). Egervary was probably the first author dealing with a system (albeit only the homogeneous one). Several methods for the nonhomogeneous system have recently been proposed based mainly on reduction to canonical forms.

We recall some results from number theory. Let a and b be integers. If there is integer γ so that $b = \gamma a$ then we say that a divides b and write $a|b$, otherwise we write $a \nmid b$. If a_1, \dots, a_n are integers, not all being zero, then the greatest common divisor (gcd) of these numbers is the greatest positive integer δ which divides all a_i , $i = 1, \dots, n$ and we write $\delta = gcd(a_1, \dots, a_n)$. We note that $\delta \geq 1$ and that δ can be written as an integer linear combination of the a_i , i.e. $\delta = z^T a$ for some $z \in R^n$. One can show that δ is the least positive integer for which the equation $a_1 x_1 + \dots + a_n x_n = \delta$ has an integer solution. Now δ plays a main role in the following

Fundamental Theorem of the Linear Diophantine Equation

Let a_1, \dots, a_n and b be integer numbers. Then the Diophantine linear equation $a_1 x_1 + \dots + a_n x_n = b$ has integer solutions if and only if $gcd(a_1, \dots, a_n) | b$. In such a case if $n > 1$ then there is an infinite number of integer solutions.

In order to find the general integer solution of the Diophantine equation $a_1 x_1 + \dots + a_n x_n = b$, the main step is to solve $a_1 x_1 + \dots + a_n x_n = \delta$, where $\delta = gcd(a_1, \dots, a_n)$, for a special integer solution. There exist several algorithms for this problem. The basic step is the computation of δ and z , often done using the algorithm of Rosser (1941), which avoids a too rapid growth of the intermediate integers, and which terminates in polynomial time, as shown by Schrijver (1986). The scaled ABS algorithm can be applied to Diophantine equations via a special choice of its parameters, originating from the following considerations and Theorems.

Suppose x_i is an integer vector. Since $x_{i+1} = x_i - \alpha_i p_i$, then x_{i+1} is integer if α_i and p_i are integers. If $v_i^T A p_i | (v_i^T r_i)$, then α_i is an integer. If H_i and z_i are respectively an integer matrix and an integer vector, then $p_i = H_i^T z_i$ is also an integer vector. Assume H_i is an integer matrix. From (6), if $v_i^T A H_i^T w_i$ divides all the components of $H_i A^T v_i$, then H_{i+1} is an integer matrix.

Conditions for the existence of an integer solution and determination of all integer solutions of the Diophantine system are given in the following theorems, generalizing the Fundamental Theorem, see Esmaeili, Mahdavi-Amiri and Spedicato (1999), or Fodor (1999) for a different proof under somewhat less general conditions.

THEOREM 1 *Let A be full rank and suppose that the Diophantine system (13) is integerly solvable. Consider the Abaffians generated by the scaled ABS algorithm with the parameter choices: H_1 is unimodular (i.e. both H_1 and H_1^{-1} are integer matrices); for $i = 1, \dots, m$, w_i is such that $w_i^T H_i A^T v_i = \delta_i$, $\delta_i = gcd(H_i A^T v_i)$. Then the following properties are true:*

- (a) *the Abaffians generated by the algorithm are well-defined and are integer matrices*

- (b) if y is a special integer solution of the first i equations, then any integer solution x of such equations can be written as $x = y + H_{i+1}^T q$ for some integer vector q .

THEOREM 2 *Let A be full rank and consider the sequence of matrices H_i generated by the scaled ABS algorithm with parameter choices as in Theorem 1. Let x_1 in the scaled ABS algorithm be an arbitrary integer vector and let z_i be such that $z_i^T H_i A^T v_i = \gcd(H_i A^T v_i)$. Then system (13) has integer solutions iff $\gcd(H_i A^T v_i)$ divides $v_i^T r_i$ for $i = 1, \dots, m$.*

We can now state the scaled ABS algorithm for Diophantine equations.

The ABS Algorithm for Diophantine Linear Equations

- (1) Choose $x_1 \in \mathbb{Z}^n$, arbitrary, $H_1 \in \mathbb{Z}^{n \times n}$, arbitrary unimodular. Let $i = 1$.
- (2) Compute $\tau_i = v_i^T r_i$ and $s_i = H_i A^T v_i$.
- (3) If ($s_i = 0$ and $\tau_i = 0$) then let $x_{i+1} = x_i$, $H_{i+1} = H_i$, $r_{i+1} = r_i$ and go to step (5) (the i th equation is redundant). If ($s_i = 0$ and $\tau_i \neq 0$) then Stop (the i th equation and hence the system is incompatible).
- (4) $\{s_i \neq 0\}$ Compute $\delta_i = \gcd(s_i)$ and $p_i = H_i^T z_i$, where $z_i \in \mathbb{Z}^n$ is an arbitrary integer vector satisfying $z_i^T s_i = \delta_i$. If $\delta_i \nmid \tau_i$ then Stop (the system is integrally inconsistent), else Compute $\alpha_i = \tau_i / \delta_i$, let $x_{i+1} = x_i - \alpha_i p_i$ and update H_i by $H_{i+1} = H_i - \frac{H_i A^T v_i w_i^T H_i}{w_i^T H_i A^T v_i}$ where $w_i \in \mathbb{R}^n$ is an arbitrary integer vector satisfying $w_i^T s_i = \delta_i$.
- (5) If $i = m$ then Stop (x_{m+1} is a solution) else let $i = i + 1$ and go to step (2).

It follows from Theorem 1 that if there exists a solution for the system (13), then $x = x_{m+1} + H_{m+1}^T q$, with arbitrary $q \in \mathbb{Z}^n$, provides all solutions of (13).

Egervary's algorithm for homogeneous Diophantine systems corresponds to the choices $H_1 = I$, $x_1 = 0$ and $w_i = z_i$, for all i . Egervary claimed, without proof, that any set of $n - m$ linearly independent rows of H_{m+1} form an integer basis for the general solution of the system. We have shown by a counterexample that Egervary's claim is not true in general; we have also provided an analysis of conditions under which m rows in H_{m+1} can be eliminated.

3. – The generalized implicit LU subclass

The generalized implicit LU (GILU) subclass is defined by taking $v_i = z_i = w_i = e_i$ and H_1 arbitrary nonsingular. The GILU subclass is well defined iff the matrix $AH_1^T E_m$ is strongly nonsingular, where $E_m = (e_1, \dots, e_m)$.

The well definiteness condition involves the matrix $AH_1^T = (H_1 a_1, \dots, H_1 a_m)^T$. H_1^T can be interpreted as a right scaling or right conditioning operator on A , acting in the same way on the different rows of A . If A is full rank but not regular, the well definiteness condition can be satisfied by simply taking H_1 as a suitable permutation matrix. By this choice LU factorization with column pivoting is imbedded in the GILU subclass. It can also be shown that all sequences x_i generated by the basic ABS class can be obtained a suitable choice of H_1 in the GILU subclass.

The given parameter choices imply the following structure for the Abaffian,

$$H_{i+1} = \begin{bmatrix} 0_{i,n} \\ S_{n-i,n} \end{bmatrix} \quad (14)$$

where $S_{n-i,n} \in R^{n-i,n}$ is full rank. The total number of multiplications is no more than $n^3 + O(n^2)$ for $m = n$, a substantial saving over algorithms in the basic ABS class which may require $3n^3$ multiplications, after the parameters are given.

The GILU subclass is related to a representation of the ABS class given in Abaffy and Spedicato (1989) in terms of $n - i$ vectors as a generalization of a method proposed by Sloboda (1978). In such a representation one takes $w_i = z_i$ and H_1 arbitrary, assuming that the feasible parameters z_i 's are known initially. Then the search vector can be written in the form $p_i = u_i^i$ where $u_j^1 = H_1^T z_j$, $j = 1, \dots, n$ and the vectors u_j^i are updated, for $i = 1, \dots, m$, by $u_j^{i+1} = u_j^i - (a_i^T u_j^i / a_i^T u_i^i) u_i^i$.

The relation between the GILU subclass and the representation in terms of $n - i$ vectors is given by the following

THEOREM 3 *Define the matrix $U_i = (u_1, \dots, u_n)$ by $u_k = 0$ for $k < i$, $u_k = u_k^i$ for $k \geq i$. Then $U_i = H_i^T$, $u_i^i = p_i$ where H_i , p_i are respectively the i -th Abaffian and search vector of the GILU subclass with initial Abaffian $H_1' = Z^T H_1$, $Z = (z_1, \dots, z_n)$.*

Further analysis shows that the matrix H_1 needs not be given explicitly at the initial step of the algorithm. The i -th row of H_1 can be given just at the i -th step. It can be considered as a vector parameter (right scaling parameter), arbitrary save that the matrix $A_i^T (H_1^T)_i$ must be nonsingular, where $(H_1^T)_i$ is the matrix comprising the first i columns of H_1^T . Therefore this formulation shows that all right preconditioners can be imbedded in the ABS class, right preconditioning being just equivalent to a change in the initial matrix H_1 .

4. – ABS Methods for KT Equations.

The KT (Kuhn-Tucker) equations are the following special linear system, related to the optimality conditions when minimizing a quadratic function with Hessian $G \in R^{n,n}$ subject to the linear equality constraint $Cp = c$, $C \in R^{m,n}$, $p, g \in R^n$, $c, z \in R^m$

$$\begin{bmatrix} G & C^T \\ C & 0 \end{bmatrix} \begin{pmatrix} p \\ z \end{pmatrix} = \begin{pmatrix} g \\ c \end{pmatrix} \quad (15)$$

If G is nonsingular, the coefficient matrix is nonsingular iff $CG^{-1}C^T$ is nonsingular. Usually G is nonsingular, symmetric and positive definite, but this assumption, required by several classical solvers, is not necessary for the ABS solvers.

To derive ABS methods using the structure of system (15), observe that (15) is equivalent to the two subsystems

$$Gp + C^T z = g \quad (16)$$

$$Cp = c. \quad (17)$$

Consider the general solution of $Cp = c$ in the ABS form, with $q \in R^n$ arbitrary

$$p = p_{m+1} + H_{m+1}^T q \quad (18)$$

The parameters used to construct p_{m+1} and H_{m+1} are arbitrary, hence (18) defines a class of algorithms.

Since the KT equations have a unique solution, there is a q which makes p the unique n -dimensional subvector defined by the first n components of the solution of (15). By multiplying $Gp + C^T z = g$ on the left by H_{m+1} we obtain the equation

$$H_{m+1}Gp = H_{m+1}g \quad (19)$$

which does not contain z . Now there are two possibilities for determining p :

(A1) Consider the system formed by (19) and (17). Such a system is solvable but overdetermined. Since $\text{rank}(H_{m+1}) = n - m$, m equations are recognized as dependent and are eliminated in step (B) of any ABS algorithm applied to this system, which then computes the unique solution.

(A2) In equation (19) replace p by the general solution (18) to give

$$H_{m+1}GH_{m+1}^T q = H_{m+1}g - H_{m+1}Gp_{m+1}. \quad (20)$$

The above system can be solved by any ABS method for a particular solution q , m equations being again removed at step (B) of the ABS algorithm as linearly dependent.

Once p is determined, one can determine z in two ways, namely:

(B1) Solve by any ABS method the overdetermined compatible system

$$C^T z = g - Gp \quad (21)$$

by removing at step (B) of the ABS algorithm the $n - m$ dependent equations.

(B2) Let $P = (p_1, \dots, p_m)$ be the matrix whose columns are the search vectors generated on the system $Cp = c$. Now $CP = L$, with L nonsingular lower diagonal. Multiplying equation (21) on the left by P^T we obtain a triangular system, defining z uniquely

$$L^T z = P^T g - P^T Gp. \quad (22)$$

Extensive numerical testing has evaluated the accuracy of the above ABS algorithms for KT equations for certain choices of the ABS parameters (corresponding to the implicit LU algorithm with row pivoting and the modified Huang algorithm). The methods have been tested against the method of Aasen and methods using the LU and the QR factorization. The experiments have shown that some ABS methods are the most accurate, in both residual and solution error; moreover some ABS algorithms are cheaper in storage and in overhead, up to one order, especially for the case when m is close to n . In particular two methods based upon the implicit LU algorithm not only have turned out to be more accurate, especially in residual error, than the method of Aasen and the method using QR factorization via Houselder matrices, but are also cheaper in number of operations (the method of Aasen has a lower storage for small m but a higher storage for large m).

In many interior point methods the main computational cost is to compute the solution for a sequence of KT problems where only G , which is diagonal, changes. In such a case the ABS methods, which initially work on the matrix C , which is unchanged, have an advantage, particularly when m is large, where the dominant cubic term decreases with m and disappears for $m = n$, so that the overhead is dominated by second order terms. Again numerical experiments show that some ABS methods are more accurate than the classical ones.

5. – A class of ABS methods for matrix equations

It is common, in particular in optimization, to find systems of matrix equations

$$A^i \bullet X = b_i, \quad i = 1, \dots, m \quad (23)$$

where operation \bullet is defined by $A \bullet B = \text{tr}(A^T B)$. We can write systems (23), with obvious definition of \circ , as

$$\mathcal{A} \circ X = b \quad (24)$$

where \mathcal{A} has the following form, with $A^i \in R^{n,n}$ for $i = 1, \dots, m$

$$\mathcal{A} = \begin{bmatrix} A^1 \\ \vdots \\ A^m \end{bmatrix}. \quad (25)$$

Problem (24) is a linear system in the space of matrices and the associated projection operators are matrices whose elements are matrices in $R^{n,n}$. This observation led us to consider a linear space denoted by $(R^{n,n})^{n,n}$ and study its linear algebra. The isomorphisms between $R^{n,n}$ and R^{n^2} , between $(R^{n,n})^{n,n}$ and R^{n^2, n^2} allow to establish ABS algorithms to solve (24), to generalize the Huang algorithm and implicit LU algorithm and to define other special algorithms. Quasi-Newton matrices satisfying linear relations (for example, symmetry and sparsity) can be described in the considered matrix form and can be solved by the proposed matrix ABS algorithm.

The ABS method for finding a solution $X \in R^{n,n}$, of system (24) is as follows, the symbol $*$ indicating transposition in the matrix space.

The matrix ABS algorithm

Step 1 Give $X^1 \in R^{n,n,n}$, $\mathcal{H}^1 \in (R)^{n,n}$, set $k = 1$.

Step 2 Compute $\tau_k = A^k \bullet X^k - b_k$ and $S^k = \mathcal{H}^k \circ A^k$.

Step 3 If $S^k \neq 0$ go to Step 4; if $S^k = 0$ and $\tau = 0$ set $X^{k+1} = X^k$, $\mathcal{H}^{k+1} = \mathcal{H}^k$ and go to Step 7 if $k < m$; otherwise stop. If $S^k = 0$ and $\tau \neq 0$ stop, the system (24) is incompatible.

Step 4 Compute $P^k \in R^{n,n}$ by

$$P^k = (\mathcal{H}^k)^* \circ Z^k \quad (26)$$

where $Z^k \in R^{n,n}$ is arbitrary save that $A^k \bullet P^k \neq 0$

Step 5 Update the approximation of the solution by

$$X^{k+1} = X^k - \alpha_k P^k, \quad \alpha_k = \tau_k / A^k \bullet P^k \quad (27)$$

If $k = m$ stop; X^{m+1} solves the system.

Step 6 Update the matrix \mathcal{H}^k by

$$\mathcal{H}^{k+1} = \mathcal{H}^k - \mathcal{H}^k \circ A^k \otimes [(\mathcal{H}^k)^* \bullet W^k] / W^k \bullet (\mathcal{H}^k \circ A^k) \quad (28)$$

where $W^k \in R^{n,n}$ is arbitrary save that $W^k \bullet (\mathcal{H}^k \circ A^k) \neq 0$

Step 7 Increment the index k by one and goto Step 2.

Properties of the matrix ABS method generalize properties of the scaled ABS class, albeit proofs are not always obvious, see Spedicato, Xia and Zhang (1999). We just recall that the general solution of (24) can be expressed as follows, with $W \in R^{n,n}$ arbitrary

$$X = X^{m+1} + (\mathcal{H}^{m+1})^* \circ W \quad (29)$$

There are natural generalizations of the Huang and the implicit LU algorithms in the ABS class. Huang matrix algorithm allows to construct solutions to the quasi-Newton equation

$$B'\delta = r \quad (30)$$

where $\delta = x' - x$ and $r = F(x') - F(x)$ when solving the nonlinear system of equations $F(x) = 0$, $r = \nabla f(x') - \nabla f(x)$ when minimizing the unconstrained function $f(x)$. Equation (30) can be solved also under the additional condition that some elements of the solution take prescribed values, by which way we can introduce the conditions of sparsity, symmetry and positive definiteness.

6. – A class of ABS derived iterative methods

When n is large, both storage and number of operations may be too large for an implementation of ABS methods using explicitly the Abaffians. One can develop methods with lower storage and operations by working with formulations of ABS methods that use vectors and then using restart or truncation. This approach leads to loss of termination and generates iterative methods. Here we describe one such class, which is derived from the following formulation of ABS methods in terms of k vectors at the k -th step, due originally to Bodon, see Abaffy and Spedicato (1989), and where we take z_k, w_k to be multiple of each other.

The Bodon-ABS vector algorithm

Let $x_1 \in R^n$ be arbitrary. Let $H_1 \in R^{n,n}$ and $V = (v_1, \dots, v_n) \in R^{n,n}$ be arbitrary nonsingular.

For $k = 1$ to n

$$\tau_k = v_k^T A x_k - v_k^T b$$

If $k > 1$ then

```

 $p_k^1 = H_1^T z_k$ 
For  $j = 1$  to  $k - 1$ 
     $p_k^{j+1} = p_k^j - \left( v_j^T A p_k^j / v_j^T A p_j \right) p_j$ 
End
 $p_k = p_k^k$ 
Else
     $p_k = H_1^T z_1$ 
Endif
 $\alpha_k = \tau_k / (v_k^T A p_k)$ 
 $x_{k+1} = x_k - \alpha_k p_k$ 
End

```

The above algorithm leads to iterative methods either via restart or via truncation. Here we consider only truncation. If m is the number of available storage vectors, then we keep only information from the last m iterations, i.e. we replace all iterations from $j = 1$ to $k - 1$ by iterations from $j = k - m$ to $k - 1$. Strategies where the kept vectors are not necessarily the last m vectors may of course be considered. The matrix H_1 should require low storage, hence we take $H_1 = I$. Parameters v_k should be linearly independent and parameters z_k feasible (no division by zero).

Algorithm ABS(m): the truncated Bodon-ABS vector algorithm

```

Let  $x_1 \in R^n$  be arbitrary.
Give the integer  $m, 1 \leq m \leq n$ .
Do  $k = 1, 2, \dots$ , until convergence
     $\tau_k = v_k^T A x_k - v_k^T b$ 
    If  $k > 1$  then
         $t = \max(1, k - m)$ 
         $p_k^t = z_k$ 
        For  $j = t$  to  $k - 1$ 
             $p_k^{j+1} = p_k^j - \left( v_j^T A p_k^j / v_j^T A p_j \right) p_j$ 
        End
         $p_k = p_k^k$ 
    Else
         $p_k = z_1, (z_1 \in R^n)$ 
    Endif
     $\alpha_k = \tau_k / (v_k^T A p_k)$ 
     $x_{k+1} = x_k - \alpha_k p_k$ 
End do

```

If $m = 1$ one can show that the truncated implicit LU, Huang and implicit QR algorithm generate respectively the Gauss-Seidel, the Kaczmarz and the De la Garza methods. For a comparison of storage and operations requirements with other well-know iterative methods see Spedicato and Li (1999).

One can prove that algorithm $\text{ABS}(m)$ is well-defined if, letting $V_k = (v_t, \dots, v_k)$ and $Z_k = (z_t, \dots, z_k)$, where $t = \max(1, k - m)$, then $V_k^T A Z_k$ is strongly non-singular for all k .

Without loss of generality, we can define the scaling parameters by

$$v_k = A^{-T} Y p_k \quad (31)$$

where Y is a symmetric, positive definite matrix. The following choices for Y define, in the original scaled ABS class, the three subclasses considered in section 1 and require a low storage for large sparse systems: $Y = I$ (the optimally scaled subclass); $Y = A^T A$ (the orthogonally scaled subclass); $Y = A^T > 0$, for $A = A^T > 0$, (the conjugate direction subclass).

Algorithm $\text{ABS}(m)$ with the parameter choice (31) has variational properties, related to those of the original ABS class, namely that x_{k+1} minimizes the error Y -weighted Euclidean norm over a linear variety spanned by the last m search vectors.

One can also show that the truncated generalized conjugate direction algorithm of Dennis and Turner (1987) can be obtained by special choices of the parameters in Algorithm $\text{ABS}(m)$. From this equivalence it follows that $\text{ORTHOMIN}(m)$, $\text{ORTHODIR}(m)$ and $\text{GMRES}(m)$ are special cases of $\text{ABS}(m)$.

The convergence of $\text{ABS}(m)$ at a linear rate can be proved by requiring that the angle between p_k and the gradient \bar{r}_k of the Y -weighted error norm be uniformly bounded away from 90° .

THEOREM 4 . *Suppose that there exists $\gamma > 0$ such that for the search vectors generated by Algorithm $\text{ABS}(m)$ with choice (31) one has*

$$|p_k^T \bar{r}_k| \geq \gamma \|p_k\|_2 \|\bar{r}_k\|_2 \quad (32)$$

for all k . Then the sequence x_k converges to x^ and satisfies*

$$\|x_{k+1} - x^*\|_Y \leq \left(1 - \frac{\gamma^2}{\text{Cond}(Y)}\right)^{1/2} \|x_k - x^*\|_Y, \quad (33)$$

where $\text{Cond}(Y)$ is defined by the ratio of the largest to the smallest eigenvalues of Y .

7. – ABSPACK and its numerical performance

The ABSPACK project aims at producing a mathematical package for solving linear and nonlinear systems and optimization problems using the ABS algorithms. The project will take several years for completion, in view of the substantial work needed to test the alternative ways ABS methods can be implemented (via different linear algebra formulations of the process, different possibilities of reprojections, different possible block formulations etc.) and of the necessity of comparing the produced software with the established packages in the market (e.g. MATLAB, LINPACK, LAPACK, UFO ...). It is expected that the software will be documented in a forthcoming monograph and will be made available to general users.

At the present state of the work FORTRAN 77 implementations have been made of several versions of the following ABS algorithms for solving linear systems:

1. The Huang and the modified Huang algorithms in two different linear algebra versions of the process, for solving determined, underdetermined and overdetermined systems, for a solution of least Euclidean norm
2. The implicit LU and implicit LX algorithm for determined, underdetermined and overdetermined linear systems, for a solution of basic type
3. The implicit QR algorithm for determined, underdetermined and overdetermined linear systems, for a solution of basic type
4. The above algorithms for some structured problems (including banded and block angular matrices)
5. An ABS algorithm proposed by Adib and Mahdavi-Amiri (1999) equivalent to a block-two ABS method
6. ABS versions of the GMRES method, some requiring less storage
7. Several ABS methods for KT equations.

For a full presentation of the above methods and their comparison with NAG, LAPACK, LINPACK and UFO codes see Bodon, Luksan and Spedicato (2000), Bodon and Spedicato (2000a,b). Some results are presented in the Appendix. There the columns refer respectively to: the problem, the dimension, the algorithm, the relative solution error (in Euclidean norm), the relative residual error in Euclidean norm (i.e. ratio of residual norm over norm of right hand side), the computed rank and the time in seconds. Computations have been performed in double precision on a Digital Alpha workstation with machine zero about 10^{-17} . All test problems have been generated with integer entries or powers of two

such that all entries are exactly represented in the machine and the right hand side can be computed exactly, so that the given solution is an exact solution of the problem as it is represented in the machine. Comparison is given with some LAPACK and LINPACK codes, including those based upon singular value decomposition (*svd*) and rank revealing QR factorization (*gqr*).

Analysis of all obtained results indicates:

1. Modified Huang is generally the most accurate ABS algorithm and compares in accuracy with the best LAPACK solvers based upon singular value factorization and rank revealing QR factorization; also the estimated ranks are usually the same.
2. On problems whose numerical estimated rank is much less than the dimension, one of the versions of modified Huang is much faster than the LAPACK codes using SVD or rank revealing QR factorization, even more than a factor 100. This is due to the fact that once an equation is recognized as dependent it does not contribute to the general overhead in ABS algorithms.
3. Modified Huang is generally faster and more accurate than other ABS methods and classical methods on KT equations.

It should be noted that the performance of the considered ABS algorithms in term of times could be improved by developing block versions, as it is the case for the LAPACK codes, a work presently in progress.

8. – Final remarks

Additional work done in the framework of the project, not described here in detail, has involved the following topics.

- Improvement of the performance of Newton method via a special truncation approach, see Deng and Wang (1998).
- Critical review of variable metric methods for unconstrained optimization with discussion of the ABS applications in this field, see Luksan and Spedicato (1998).
- Analysis of the relations between the ABS methods and the classical method of averaging functional corrections, see Gredzhuk and Petrina (1998).
- Development of indefinitely preconditioned truncated Newton methods for large sparse equality constrained nonlinear programming problems, see Luksan and Vlcek (1998).
- Computation and update of inertias of KKT matrices for use in quadratic programming, see Zhang (1999).
- Further applications of the implicit LX algorithm to the simplex method for the LP problem, see Spedicato and Xia (1999).

The field of ABS methods is now mature from a theoretical point of view, albeit there are exciting possibilities for applications to new fields, e.g. the eigenvalue problems. We expect that the completion of the project ABSPACK will provide a useful new instrument for users of mathematical software.

REFERENCES

- [1] Abaffy J. and Spedicato E., *ABS Projection Algorithms: Mathematical Techniques for Linear and Nonlinear Equations*, Ellis Horwood, Chichester, 1989
- [2] Abaffy J., Broyden C.G. and Spedicato E., A class of direct methods for linear systems, *Numerische Mathematik* 45, 361-376, 1984
- [3] Adib M. and Mahdavi-Amiri N., A new ABS-based approach for solving linear systems with a rank two update of the Abaffian, preprint, Sharif University of Technology, Tehran, 1999

- [4] Bertrand I. and Betti G., *Traité élémentaire d' algebre*, 1850
- [5] Bodon E. and Spedicato E., Numerical experiments with block ABS methods for linear systems, to appear, 2000a
- [6] Bodon E. and Spedicato E., Numerical experiments with the ABS-GMRES method, to appear, 2000b
- [7] Bodon E., Luksan L. and Spedicato E., Numerical performance of an ABS package for linear systems, to appear, 2000
- [8] Deng N. and Wang Z., Newton method can be beaten, Report DMSIA 98/9, University of Bergamo, 1998
- [9] Dennis J. and Turner K., Generalized conjugate directions, *Linear Algebra and its Applications* 88/89, 187-209, 1987
- [10] Egervary E., Aufloesung eines homogenen linearen diophantischen Gleichungssystems mit Hilf von Projektormatrizen, *Publ. Math. Debrecen* 4, 481-483, 1955
- [11] Esmaeili H., Mahdavi-Amiri and Spedicato E., Solution of Diophantine linear systems via the ABS methods, Report DMSIA 99/29, University of Bergamo, 1999 (submitted to *Numerische Mathematik*)
- [12] Fodor S., A class of ABS methods for Diophantine systems of equations, Part I, Preprint, Computer Science Department, University of Economics, Budapest, 1999
- [13] Gredzhuk V. and Petrina D., The ABS method and the method of averaging functional corrections, Report DMSIA 98/8, University of Bergamo, 1998
- [14] Luksan L. and Spedicato E., Variable metric methods for unconstrained optimization, Report DMSIA 98/7, University of Bergamo, 1998
- [15] Luksan L. and Vlcek J., Indefinitely preconditioned truncated Newton method for large sparse equality constrained nonlinear programming problems, Report DMSIA 98/5, University of Bergamo, 1998
- [16] Mirnia K.: Numerical experiments with iterative refinement of solutions of linear equations by ABS methods, Report DMSIA 32/96, University of Bergamo, 1996
- [17] Nicolai S. and Spedicato E.: A bibliography of the ABS methods, *OMS* 8, 171-183, 1997
- [18] Rosser J.B., A note on the linear Diophantine equation, *Amer. Math. Monthly* 48, 662-666, 1941

- [19] Schrijver A., Theory of Linear and Integer Programming, John Wiley and Sons, 1986
- [20] Sloboda F., A parallel projection method for linear algebraic systems, *Aplic. Mat. Ceskosl. Akad. Ved.* 23, 185-198, 1978
- [21] Spedicato E. and Li Z., On some classes of iterative ABS-like methods for large scale linear systems, 1999 (to appear)
- [22] Spedicato E. and Xia Z., ABS formulation of Fletcher implicit LU method for factorizing a matrix and solving linear equations and LP problems, Report DMSIA 99/1, University of Bergamo, 1999
- [23] Spedicato E., Xia Z. and Zhang L., The implicit LX method of the ABS class, *Optimization Methods and Software* 8, 99-110, 1997
- [24] E. Spedicato, Z. Xia and L. Zhang, Solving linear matrix equations via the ABS class and applications to Quasi-Newton equation, Report DMSIA 12/99, 1999, (submitted to *Linear Algebra and its Applications*)
- [25] Zhang L., Computing inertias of KKT matrix and reduced Hessian via the ABS algorithm for applications to quadratic programming, Report DMSIA 99/7, University of Bergamo, 1999
- [26] Zhang L., Xia Z. and Feng E.: Introduction to ABS methods in optimization, Dalian University of Technology Press, 1999 (in Chinese)

APPENDIX

RESULTS ON DETERMINED LINEAR SYSTEMS

Condition number: 0.21D+20

IDF2	2000	huang2	0.10D+01	0.69D-11	2000	262.00
IDF2	2000	mod.huang2	0.14D+01	0.96D-12	4	7.00
IDF2	2000	lu lapack	0.67D+04	0.18D-11	2000	53.00
IDF2	2000	qr lapack	0.34D+04	0.92D-12	2000	137.00
IDF2	2000	gqr lapack	0.10D+01	0.20D-14	3	226.00
IDF2	2000	lu linpack	0.67D+04	0.18D-11	2000	136.00

Condition number: 0.10D+61

IR50	1000	huang2	0.46D+00	0.33D-09	1000	36.00
IR50	1000	mod.huang2	0.46D+00	0.27D-14	772	61.00
IR50	1000	lu lapack	0.12D+04	0.12D+04	972	7.00
IR50	1000	qr lapack	0.63D+02	0.17D-12	1000	17.00
IR50	1000	gqr lapack	0.46D+00	0.42D-14	772	29.00
IR50	1000	lu linpack	--- break-down ---			

RESULTS ON OVERDETERMINED SYSTEMS

Condition number: 0.16D+21

IDF3	1050	950	huang7	0.32D+04	0.52D-13	950	31.00
IDF3	1050	950	mod.huang7	0.14D+04	0.20D-09	2	0.00
IDF3	1050	950	qr lapack	0.37D+13	0.83D-02	950	17.00
IDF3	1050	950	svd lapack	0.10D+01	0.24D-14	2	145.00
IDF3	1050	950	gqr lapack	0.10D+01	0.22D-14	2	27.00

Condition number: 0.63D+19

IDF3	2000	400	huang7	0.38D+04	0.35D-12	400	9.00
IDF3	2000	400	mod.huang7	0.44D+03	0.67D-12	2	0.00
IDF3	2000	400	impl.qr5	0.44D+03	0.62D-16	2	0.00
IDF3	2000	400	expl.qr	0.10D+01	0.62D-03	2	0.00
IDF3	2000	400	qr lapack	0.45D+12	0.24D-02	400	8.00
IDF3	2000	400	svd lapack	0.10D+01	0.65D-15	2	17.00
IDF3	2000	400	gqr lapack	0.10D+01	0.19D-14	2	12.00

RESULTS ON UNDERDETERMINED LINEAR SYSTEMS

Condition number: 0.29D+18							
IDF2	400	2000	huang2	0.12D-10	0.10D-12	400	12.00
IDF2	400	2000	mod.huang2	0.36D-08	0.61D-10	3	1.00
IDF2	400	2000	qr lapack	0.29D+03	0.37D-14	400	9.00
IDF2	400	2000	svd lapack	0.43D-13	0.22D-14	3	68.00
IDF2	400	2000	gqr lapack	0.18D-13	0.24D-14	3	12.00
Condition number: 0.24D+19							
IDF3	950	1050	huang2	0.00D+00	0.00D+00	950	33.00
IDF3	950	1050	mod.huang2	0.00D+00	0.00D+00	2	1.00
IDF3	950	1050	qr lapack	0.24D+03	0.56D-14	950	17.00
IDF3	950	1050	svd lapack	0.17D-14	0.92D-16	2	178.00
IDF3	950	1050	gqr lapack	0.21D-14	0.55D-15	2	26.00

RESULTS ON KT SYSTEMS

Condition number: 0.26D+21

IDF2	1000	900	mod.huang	0.55D+01	0.23D-14	16	24.00
IDF2	1000	900	impl.lu8	0.44D+13	0.21D-03	1900	18.00
IDF2	1000	900	impl.lu9	0.12D+15	0.80D-02	1900	21.00
IDF2	1000	900	lu lapack	0.25D+03	0.31D-13	1900	62.00
IDF2	1000	900	range space	0.16D+05	0.14D-11	1900	87.00
IDF2	1000	900	null space	0.89D+03	0.15D-12	1900	93.00

Condition number: 0.70D+20

IDF2	1200	600	mod.huang	0.62D+01	0.20D-14	17	36.00
IDF2	1200	600	impl.lu8	0.22D+07	0.10D-08	1800	44.00
IDF2	1200	600	impl.lu9	0.21D+06	0.56D-09	1800	33.00
IDF2	1200	600	lu lapack	0.10D+03	0.79D-14	1800	47.00
IDF2	1200	600	range space	0.11D+05	0.15D-11	1800	63.00
IDF2	1200	600	null space	0.38D+04	0.13D-12	1800	105.00